Exercise 9

Solve the differential equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 1 + e^{-2x}$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2y_c}{dx^2} - \frac{dy_c}{dx} - 6y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad \frac{dy_c}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y_c}{dx^2} = r^2e^{rx}$$

Substitute these formulas into the ODE.

$$r^2e^{rx} - re^{rx} - 6(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - r - 6 = 0$$

Solve for r.

$$(r-3)(r+2) = 0$$

$$r = \{-2, 3\}$$

Two solutions to the ODE are e^{-2x} and e^{3x}. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$y_c(x) = C_1e^{-2x} + C_2e^{3x}$$

C_1 and C_2 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2y_p}{dx^2} - \frac{dy_p}{dx} - 6y_p = 1 + e^{-2x} \tag{3}$$

Since the inhomogeneous term is a polynomial of degree 0 and an exponential, the particular solution would be $y_p = A + Be^{-2x}$. e^{-2x} is already part of y_c, though, so an extra factor of x on the exponential is needed: $y_p = A + Bxe^{-2x}$.

$$y_p = A + Bxe^{-2x} \quad \rightarrow \quad \frac{dy_p}{dx} = Be^{-2x} - 2Bxe^{-2x} \quad \rightarrow \quad \frac{d^2y_p}{dx^2} = -2Be^{-2x} - 2Be^{-2x} + 4Bxe^{-2x}$$

Substitute these formulas into equation (3).

$$(-2Be^{-2x} - 2Be^{-2x} + 4Bxe^{-2x}) - (Be^{-2x} - 2Bxe^{-2x}) - 6(A + Bxe^{-2x}) = 1 + e^{-2x}$$

Simplify the left side.

$$(-6A) + (-5B)e^{-2x} = 1 + e^{-2x}$$

Match the coefficients to get a system of equations for A and B.

$$-6A = 1$$

$$-5B = 1$$

Solving it yields

$$A = -\frac{1}{6} \quad \text{and} \quad B = -\frac{1}{5}.$$

The particular solution is then

$$\begin{aligned} y_p &= A + Bxe^{-2x} \\ &= -\frac{1}{6} - \frac{1}{5}xe^{-2x}. \end{aligned}$$

Therefore, the general solution to the original ODE is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1e^{-2x} + C_2e^{3x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}. \end{aligned}$$